

# Kähler geometry and SUSY mechanics.

Stefano Bellucci<sup>a</sup> and Armen Nersessian<sup>a b c</sup>

<sup>a</sup>INFN, Laboratori Nazionali di Frascati, P.O. Box 13, I-00044, Frascati, Italy

<sup>b</sup>JINR, Bogolyubov Laboratory of Theoretical Physics, Dubna, 141980 Russia

<sup>c</sup>Yerevan State University, A.Manoojian, 1, Yerevan, 375025 Armenia

We present two examples of SUSY mechanics related with Kähler geometry. The first system is the  $N = 4$  supersymmetric one-dimensional sigma-model proposed in [hep-th/0101065](#). Another system is the  $N = 2$  SUSY mechanics whose phase space is the external algebra of an arbitrary Kähler manifold. The relation of these models with antisymplectic geometry is discussed.

## 1. Introduction

Supersymmetric mechanics attracts permanent interest since its introduction [1]. However, studies focussed mainly on the  $N = 2$  case, and the most important case of  $N = 4$  mechanics did not receive enough attention, though some interesting observations were made about this subject: let us mention that the most general  $N = 4, D = 1, 3$  supersymmetric mechanics described by real superfield actions were studied in Refs. [2,3] respectively, and those in arbitrary  $D$  in Ref. [4]; in [5]  $N = 4, D = 2$  supersymmetric mechanics described by chiral superfield actions were considered; the general study of supersymmetric mechanics with arbitrary  $N$  was performed recently in Ref. [6]. In the Hamiltonian language classical supersymmetric mechanics can be formulated in terms of superspace equipped with some supersymplectic structure (and corresponding non-degenerate Poisson brackets). After quantization the odd coordinates are replaced by the generators of Clifford algebra. It is easy to verify that the minimal dimension of phase superspace, which allows to describe a  $D$ -dimensional supersymmetric mechanics with *nonzero* potential terms, is  $(2D|2D)$ , while supersymmetry specifies both the admissible sets of configuration spaces and potentials. In our recent paper [7] we proposed the  $N = 4$  supersymmetric one-dimensional sigma-models (with and without central charge) on Kähler manifold with  $(2d|2d)\mathbb{C}$ -

dimensional phase space. We have shown that the constructed mechanics can be obtained by dimensional reduction from  $N = 2$  supersymmetric  $(1+1)$ -dimensional sigma-models by Alvarez-Gaumé and Freedman [8]; in the simplest case of  $d = 1$  and in the absence of central charge these systems coincide with the  $N = 4$  supersymmetric mechanics described by the chiral superfield action [5].

In Section 2 we present the  $N = 4$  one-dimensional supersymmetric sigma-model constructed in Ref. [7].

In Section 3 we present a new model of  $N = 2$  supersymmetric mechanics with phase space corresponding the external algebra of a Kähler manifold. This construction seems to be the most general  $N = 2$  SUSY mechanics  $(2D|2D)$ -dimensional phase space. We also consider the relation of presented system with the antisymplectic geometry, in the context of the old problem suggested by D.V.Volkov et al. [10].

## 2. Sigma-models with $N = 4$ SUSY.

In order to get a one-dimensional  $N = 4$  supersymmetric sigma-model with  $(2D|2D)$ -dimensional phase superspace one should require that the target space  $M_0$  is a Kähler manifold  $(M_0, g_{a\bar{b}}dz^a d\bar{z}^b)$ ,  $g_{a\bar{b}} = \partial^2 K(z, \bar{z}) / \partial z^a \partial \bar{z}^b$ . This restriction follows also from the considerations of superfield actions: indeed, the  $N$ -extended supersymmetric mechan-

ics obtained from the action depending on  $D$  real superfields, have a  $(2D|ND)_{\mathbb{R}}$ -dimensional symplectic manifold, whereas those obtained from the action depending on  $d$  chiral superfields have a  $(2d|Nd/2)_{\mathbb{C}}$ -dimensional phase space, with the configuration space being a  $2d$ -dimensional Kähler manifold.

In that case the phase superspace can be equipped by the supersymplectic structure

$$\begin{aligned}\Omega &= \omega_0 - i\partial\bar{\partial}\mathbf{g} = \\ &= d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a + \\ &+ R_{a\bar{b}c\bar{d}}\eta_i^a \bar{\eta}_i^b dz^a \wedge d\bar{z}^b + g_{a\bar{b}} D\eta_i^a \wedge D\bar{\eta}_i^b\end{aligned}\quad (1)$$

where

$$\begin{aligned}\mathbf{g} &= ig_{a\bar{b}}\eta^a \sigma_0 \bar{\eta}^b, \\ D\eta_i^a &= d\eta_i^a + \Gamma_{bc}^a \eta_i^b dz^c, \quad i = 1, 2\end{aligned}\quad (2)$$

while  $\Gamma_{bc}^a$ ,  $R_{a\bar{b}c\bar{d}}$  are respectively the connection and curvature of the Kähler structure, the odd coordinates  $\eta_i^a$  belong to the external algebra  $\Lambda(M_0)$ , i. e. transforms as  $dz^a$ . This symplectic structure becomes canonical in the coordinates  $(p_a, \chi^k)$

$$\begin{aligned}p_a &= \pi_a - \frac{i}{2}\partial_a \mathbf{g}, \quad \chi_i^m = e_b^m \eta_i^b : \\ \Omega &= dp_a \wedge dz^a + d\bar{p}_a \wedge d\bar{z}^a + d\chi_i^m \wedge d\bar{\chi}_i^{\bar{m}},\end{aligned}\quad (3)$$

where  $e_a^m$  are the einbeins of the Kähler structure:  $e_a^m \delta_{m\bar{m}} \bar{e}_b^{\bar{m}} = g_{a\bar{b}}$ . So to quantize this model, one chooses

$$\hat{p}_a = -i\frac{\partial}{\partial z^a}, \quad \hat{\bar{p}}_a = -i\frac{\partial}{\partial \bar{z}^a}, \quad [\hat{\chi}_i^m, \hat{\bar{\chi}}_j^{\bar{n}}]_+ = \delta^{m\bar{n}} \delta_{ij}.$$

The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates)

$$\begin{aligned}\{\pi_a, z^b\} &= \delta_a^b, \quad \{\pi_a, \eta_i^b\} = -\Gamma_{ac}^b \eta_i^c, \\ \{\pi_a, \bar{\pi}_b\} &= -R_{a\bar{b}c\bar{d}} \eta_i^c \bar{\eta}_i^d, \quad \{\eta_i^a, \bar{\eta}_j^b\} = g^{a\bar{b}} \delta_{ij}.\end{aligned}$$

To construct on this phase superspace the Hamiltonian mechanics with standard  $N = 4$  supersymmetry algebra

$$\begin{aligned}\{Q_i^+, Q_j^-\} &= \delta_{ij} \mathcal{H}, \\ \{Q_i^{\pm}, Q_j^{\pm}\} &= \{Q_i^{\pm}, \mathcal{H}\} = 0, \quad i = 1, 2,\end{aligned}\quad (4)$$

let us choose the supercharges given by the functions

$$Q_1^+ = \pi_a \eta_1^a + iU_{\bar{a}} \bar{\eta}_2^{\bar{a}}, \quad Q_2^+ = \pi_a \eta_2^a - iU_{\bar{a}} \bar{\eta}_1^{\bar{a}}. \quad (5)$$

Then, calculating the commutators (Poisson brackets) of these functions, we get that the supercharges (5) belong to the superalgebra (4) when the functions  $U_a, \bar{U}_{\bar{a}}$  are of the form

$$U_a(z) = \frac{\partial U(z)}{\partial z^a}, \quad \bar{U}_{\bar{a}}(\bar{z}) = \frac{\partial \bar{U}(\bar{z})}{\partial \bar{z}^{\bar{a}}}, \quad (6)$$

while the Hamiltonian reads

$$\begin{aligned}\mathcal{H} &= g^{a\bar{b}}(\pi_a \bar{\pi}_b + U_a \bar{U}_{\bar{b}}) - \\ &- iU_{a;b} \eta_1^a \eta_2^b + i\bar{U}_{\bar{a};\bar{b}} \bar{\eta}_1^{\bar{a}} \bar{\eta}_2^{\bar{b}} - R_{a\bar{b}c\bar{d}} \eta_1^a \bar{\eta}_1^b \eta_2^c \bar{\eta}_2^d,\end{aligned}\quad (7)$$

where  $U_{a;b} \equiv \partial_a \partial_b U - \Gamma_{ab}^c \partial_c U$ .

The constant of motion counting the number of fermions, reads

$$\mathcal{F} = ig_{a\bar{b}} \eta^a \sigma_3 \bar{\eta}^b : \quad \{Q_i^{\pm}, \mathcal{F}\} = \pm i Q_i^{\pm}, \quad (8)$$

Notice that the above-presented  $N = 4$  SUSY mechanics for the simplest case, i.e.  $d = 1$ , was obtained by Berezovoy and Pashnev [5] from the chiral superfield action

$$\mathcal{S} = \frac{1}{2} \int K(\Phi, \bar{\Phi}) + 2 \int U(\Phi) + 2 \int \bar{U}(\bar{\Phi}) \quad (9)$$

where  $\Phi$  is a chiral superfield. It seems to be obvious that a similar action depending on  $d$  chiral superfields will generate the above-presented  $N = 4$  SUSY mechanics.

Let us consider a generalization of the above system, which possesses  $N = 4$  supersymmetry with central charge

$$\begin{aligned}\{\Theta_i^+, \Theta_j^-\} &= \delta_{ij} \mathcal{H} + \mathcal{Z} \sigma_{ij}^3, \quad \{\Theta_i^{\pm}, \Theta_j^{\pm}\} = 0, \\ \{\mathcal{Z}, \mathcal{H}\} &= \{\mathcal{Z}, \Theta_k^{\pm}\} = 0.\end{aligned}\quad (10)$$

For this purpose one introduces the supercharges

$$\begin{aligned}\Theta_1^+ &= (\pi_a + iG_{,a}(z, \bar{z})) \eta_1^a + i\bar{U}_{,\bar{a}}(\bar{z}) \bar{\eta}_2^{\bar{a}}, \\ \Theta_2^+ &= (\pi_a - iG_{,a}(z, \bar{z})) \eta_2^a - i\bar{U}_{,\bar{a}}(\bar{z}) \bar{\eta}_1^{\bar{a}},\end{aligned}\quad (11)$$

where the real function  $G(z, \bar{z})$  obeys the conditions

$$\partial_a \partial_b G + \Gamma_{ab}^c \partial_c G = 0, \quad G_{,a}(z, \bar{z}) g^{a\bar{b}} \partial_{\bar{b}} \bar{U}(\bar{z}) = 0. \quad (12)$$

The first equation in (12) is nothing but the Killing equation of the underlying Kähler structure (let us remind, that the isometries of the Kähler structure are Hamiltonian holomorphic vector fields) given by the vector

$$\mathbf{G} = G^a(z) \partial_a + \bar{G}^{\bar{a}}(\bar{z}) \bar{\partial}_{\bar{a}}, \quad G^a = ig^{a\bar{b}} \bar{\partial}_{\bar{b}} G. \quad (13)$$

The second equation means that the vector field  $\mathbf{G}$  leaves the holomorphic function invariant

$$\mathcal{L}_{\mathbf{G}}U = 0 \Rightarrow G^a(z)U_a(z) = 0.$$

Calculating the Poisson brackets of these supercharges, we get explicit expressions for the Hamiltonian

$$\begin{aligned} \mathcal{H} \equiv & g^{a\bar{b}} (\pi_a \bar{\pi}_{\bar{b}} + G_{,a} G_{\bar{b}} + U_{,a} \bar{U}_{\bar{b}}) - \\ & - iU_{a;b} \eta_1^a \eta_2^b + i\bar{U}_{\bar{a};\bar{b}} \bar{\eta}_1^{\bar{a}} \bar{\eta}_2^{\bar{b}} + \frac{1}{2} G_{a\bar{b}} (\eta_1^a \bar{\eta}_2^{\bar{b}} - \\ & - R_{a\bar{b}cd} \eta_1^a \bar{\eta}_1^{\bar{b}} \eta_2^c \bar{\eta}_2^{\bar{d}}) \end{aligned} \quad (14)$$

and the central charge

$$\mathcal{Z} = i(G^a \pi_a + G^{\bar{a}} \bar{\pi}_{\bar{a}}) + \frac{i}{2} \partial_a \bar{\partial}_{\bar{b}} G (\eta^a \sigma_3 \bar{\eta}^{\bar{b}}). \quad (15)$$

It can be checked by a straightforward calculation that the function  $\mathcal{Z}$  indeed belongs to the center of the superalgebra (10). The scalar part of each phase with standard  $N = 2$  supersymmetry can be interpreted as a particle moving on the Kähler manifold in the presence of an external magnetic field with strength  $F = iG_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$  and in the potential field  $U_{,a}(z)g^{a\bar{b}}\bar{U}_{\bar{b}}(\bar{z})$ .

Assuming that  $(M_0, g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}})$  is the hyper-Kähler metric and that  $U(z) + \bar{U}(\bar{z})$  is a tri-holomorphic function while the function  $G(z, \bar{z})$  defines a tri-holomorphic Killing vector, one should get the  $N = 8$  supersymmetric one-dimensional sigma-model. In that case instead of the phase with standard  $N = 2$  SUSY arising in the Kähler case, we shall get the phase with standard  $N = 4$  SUSY. The latter system can be viewed as a particular case of  $N = 4$  SUSY mechanics describing the low-energy dynamics of monopoles and dyons in  $N = 2, 4$  super-Yang-Mills theory [12]. Notice that, in contrast to the  $N = 4$  mechanics suggested in the mentioned papers, in the above-proposed (hypothetic) construction also the four hidden supersymmetries could be explicitly written.

The Lagrangian of the system is of the form

$$\begin{aligned} \mathcal{L} = & g_{a\bar{b}} \left( \dot{z}^a \dot{\bar{z}}^{\bar{b}} + \frac{1}{2} \eta_k^a \frac{D\bar{\eta}_k^{\bar{b}}}{d\tau} + \frac{1}{2} \frac{D\eta_k^a}{d\tau} \bar{\eta}_k^{\bar{b}} \right) - \\ & - g^{a\bar{b}} (G_a G_{\bar{b}} + U_a \bar{U}_{\bar{b}}) + \\ & + iU_{a;b} \eta_1^a \eta_2^b - i\bar{U}_{\bar{a};\bar{b}} \bar{\eta}_1^{\bar{a}} \bar{\eta}_2^{\bar{b}} + R_{a\bar{b}cd} \eta_1^a \bar{\eta}_1^{\bar{b}} \eta_2^c \bar{\eta}_2^{\bar{d}}. \end{aligned} \quad (16)$$

So, we get the Lagrangian for a one-dimensional sigma-model with four exact real supersymmetries. It can be straightforwardly obtained by

the dimensional reduction of the  $N = 2$  supersymmetric  $(1+1)$  dimensional sigma-model by Alvarez-Gaumé and Freedman [8] (the mechanical counterpart of this system without potential term was constructed in [11]).

### 3. $N = 2$ SUSY mechanics with Kähler phase space

Let us consider a supersymmetric mechanics whose phase superspace is the external algebra of the Kähler manifold  $\Lambda(M)$ , where  $(M, g_{A\bar{B}}(z, \bar{z}) dz^A d\bar{z}^{\bar{B}})$  plays the role of the phase space of underlying Hamiltonian mechanics. The phase superspace is  $(D|D)_{\mathbb{C}}$ -dimensional supermanifold equipped by the Kähler structure [13]

$$\begin{aligned} \Omega = & i\partial\bar{\partial} \left( K(z, \bar{z}) - ig_{A\bar{B}} \theta^A \bar{\theta}^{\bar{B}} \right) = \\ = & i(g_{A\bar{B}} + iR_{A\bar{B}C\bar{D}} \theta^C \bar{\theta}^{\bar{D}}) dz^A \wedge d\bar{z}^{\bar{B}} \\ & + g_{A\bar{B}} D\theta^A \wedge D\bar{\theta}^{\bar{B}}, \end{aligned} \quad (17)$$

where  $D\theta^A = d\theta^A + \Gamma_{BC}^A \theta^B dz^C$ , and  $\Gamma_{BC}^A$ ,  $R_{A\bar{B}C\bar{D}}$  are respectively the Cristoffel symbols and curvature tensor of the underlying Kähler metrics  $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$ .

The corresponding Poisson bracket can be presented in the form

$$\{ \quad, \quad \} = i\tilde{g}^{A\bar{B}} \nabla_A \wedge \bar{\nabla}_{\bar{B}} + g^{A\bar{B}} \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \bar{\theta}^{\bar{B}}} \quad (18)$$

where

$$\nabla_A = \frac{\partial}{\partial z^A} - \Gamma_{AB}^C \theta^B \frac{\partial}{\partial \theta^C},$$

and

$$\tilde{g}_{A\bar{B}}^{-1} = (g_{A\bar{B}} + iR_{A\bar{B}C\bar{D}} \theta^C \bar{\theta}^{\bar{D}}).$$

On this phase superspace one can immediately construct a mechanics with standard  $N = 2$  supersymmetry

$$\{Q_+, Q_-\} = \mathcal{H}, \quad \{Q_{\pm}, Q_{\pm}\} = \{Q_{\pm}, \mathcal{H}\} = 0, \quad (19)$$

given by the supercharges

$$Q_+^0 = \partial_A K(z, \bar{z}) \theta^A, \quad Q_-^0 = \partial_{\bar{A}} K(z, \bar{z}) \bar{\theta}^{\bar{A}} \quad (20)$$

where  $K(z, \bar{z})$  is the Kähler potential of the underlying Kähler structure, defined up to holomorphic and anti-holomorphic functions,

$$K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}).$$

The Hamiltonian of the system reads

$$\mathcal{H}_0 = g^{A\bar{B}} \partial_A K \partial_{\bar{B}} K - i g_{A\bar{B}} \theta^A \bar{\theta}^{\bar{B}} + i \theta^C K_{C;\bar{A}} \tilde{g}^{A\bar{B}} K_{\bar{B};\bar{D}} \bar{\theta}^{\bar{D}} \quad (21)$$

where  $K_{A;\bar{B}} = \partial_A \partial_{\bar{B}} K - \Gamma_{AB}^C \partial_C K$ .

One also consider another mechanics with standard  $N = 2$  SUSY whose supercharges are given by the expressions

$$Q_+^c = \partial_A G(z, \bar{z}) \theta^A, \quad Q_-^c = \partial_{\bar{A}} G(z, \bar{z}) \bar{\theta}^{\bar{A}} \quad (22)$$

where  $G(z, \bar{z})$  is the Killing potential of the underlying Kähler structure,

$$\partial_A \partial_B G - \Gamma_{AB}^C \partial_C G = 0, \\ G^A(z) = g^{A\bar{B}} \partial_{\bar{B}} G(z, \bar{z}).$$

In this case the Hamiltonian of system reads

$$\mathcal{H}^c = g_{A\bar{B}} G^A G^{\bar{B}} + i \bar{\theta}^{\bar{D}} G_{A\bar{D}} \tilde{g}^{A\bar{B}} G_{C\bar{B}} \theta^C, \quad (23)$$

where  $G_{A\bar{B}} = \partial_A \partial_{\bar{B}} G(z, \bar{z})$ .

The commutators of the supercharges in these particular examples read:

$$\{Q_\pm^c, Q_\pm^0\} = \mathcal{R}_\pm, \quad \{Q_\pm^c, Q_\mp^0\} = \mathcal{Z}, \quad (24)$$

where

$$\tilde{\mathcal{Z}} \equiv G(z, \bar{z}) + i G_{A\bar{B}}(z, \bar{z}) \theta^A \bar{\theta}^{\bar{B}}, \\ \mathcal{R}_+ = i \theta^C K_{C;\bar{A}} \tilde{g}^{A\bar{B}} G_{\bar{B};\bar{D}} \theta^{\bar{D}}, \\ \mathcal{R}_- = \bar{\mathcal{R}}_+ . \quad (25)$$

Hence, introducing the superschages

$$\Theta_\pm = Q_\pm^0 \pm i Q_\mp^c, \quad (26)$$

we can define  $N = 2$  SUSY mechanics specified by the presence of central charge  $\mathcal{Z}$ :

$$\{\Theta_+, \Theta_-\} = \tilde{\mathcal{H}}, \quad \{\Theta_\pm, \Theta_\pm\} = \pm i \mathcal{Z} \\ \{\mathcal{Z}, \Theta_\pm\} = 0, \quad -\{\tilde{\mathcal{H}}, \Theta_\mp\} = 0, \quad \{\mathcal{Z}, \tilde{\mathcal{H}}\} = 0. \quad (27)$$

The Hamiltonian of this generalized mechanics is defined by the expression

$$\tilde{\mathcal{H}} = \mathcal{H}_0 + \mathcal{H}_c + i \mathcal{R}_+ - i \mathcal{R}_-. \quad (28)$$

A “fermionic number” is of the form:

$$\tilde{\mathcal{F}} = i g_{A\bar{B}} \theta^A \bar{\theta}^{\bar{B}} : \{\tilde{\mathcal{F}}, \Theta_\pm\} = \pm i \Theta_\pm. \quad (29)$$

Choosing the Kähler manifold to be a special type of the (co)tangent bundle of some Kähler manifold, one can provide the system by the additional pair of supercharges, recovering the above-constructed model of sigma-model with standard  $N = 4$  SUSY.

The phase space of the system under consideration can be equipped, in addition to the Poisson bracket corresponding to (17), with the antibracket (odd Poisson bracket) associated with the odd Kähler potential  $K_\alpha = \alpha Q_+^0 + \bar{\alpha} Q_-^0$ ,  $\alpha = 1, i$ :

$$\{ \quad , \quad \}_1^{(\alpha)} = \alpha g^{\bar{A}B} \nabla_{\bar{A}} \wedge \frac{\partial}{\partial \theta^B} + c.c. \quad (30)$$

It is easy to observe, that the following equality holds [13]:

$$\{\tilde{\mathcal{Z}}, \quad\} = \{Q^\alpha, \quad\}_1^{(\alpha)}, \quad (31)$$

where

$$Q^\alpha = \alpha Q_+^c + \bar{\alpha} Q_-^c. \quad (32)$$

Hence, the anti-symplectic structure generated by supercharges of the  $N = 2$  SUSY mechanics (20), (21) define the pair of anti-Hamiltonian structures for the Hamiltonian field corresponding to central charge (25), while the supercharges of the SUSY mechanics (22), (23) play the role of corresponding Hamiltonians.

The following relation holds:

$$\{G + \mathcal{F}, G + \mathcal{F}\}_1^{(\alpha)} = 2Q^\alpha, \quad (33)$$

which can be interpreted as  $N = 1$  supersymmetry of the anti-Hamiltonian mechanics given by the (odd) Hamiltonian  $Q$ . However, this supersymmetry is *trivial* (or false): it does not changes the initial Hamiltonian dynamics, generated on the base manifold by the Killing potential  $G$ . On the other hand, squaring the odd Hamiltonian  $Q$  under the even Poisson bracket (corresponding to squaring the corresponding quantum-mechanical supercharge  $\hat{Q}$ ) yields the Hamiltonian (23). Hence, in the  $N = 2$  SUSY mechanics given by the Hamiltonian (23), allows us to get the “square root” by the use of the antibracket.

Notice that the supermanifolds provided by the even and odd symplectic (and Kähler) structures,

and particularly Eq. 31, were studied in connection with the problem of describing the supersymmetric mechanics in terms of antibrackets [13,14], which was considered for the first time by D.V.Volkov et al. [10]. Let us remind that until the 1980's the odd Poisson brackets (antibrackets) had no any applications in theoretical physics, due to the nontrivial Grassmann grading, and the absence of consistent quantization schemes. This situation drastically changed in 1981, when I.A.Batalin and G.A.Vilkovisky suggested a covariant Lagrangian BRST quantization scheme (which is presently known as BV-formalism) [16], whose key ingredient was the odd Poisson bracket. A bit later, in 1983, D.V.Volkov claimed, that the antibrackets can be considered, due to their non-zero Grassmann grading, as the “square root” of usual super-Poisson brackets relating the bosonic and fermionic components of super-spinors [17]. The study of the possibility of an antibracket formulation of supersymmetric Hamiltonian systems, i.e. Eq. 31, was one of the steps of that program. Later, the supersymmetric mechanics, which are Hamiltonian with respect to both even and odd Poisson brackets, were found to be useful in equivariant cohomology, e.g. for the construction of equivariant characteristic classes and the derivation of localization formulae [15]. Indeed, the vector field (31) can be identified with the Lie derivative, if choose  $\alpha = i$ :

$$\{Q, \}_1^{(i)} = iG^A(z) \frac{\partial}{\partial z^A} + iG_{,C}^A(z) \theta^C \frac{\partial}{\partial z^A} + c.c. \quad (34)$$

The vector field  $\{\tilde{\mathcal{F}}, \}_1^{(i)}$ , corresponds to the external differential

$$\{\mathcal{F}, \}_1^{(i)} = \theta^A \frac{\partial}{\partial z^A} + c.c., \quad (35)$$

while  $\{G, \}_1^{(i)}$  corresponds to the operator of inner product

$$\{G, \}_1^{(i)} = iG^A(z) \frac{\partial}{\partial \theta^A} + c.c. \quad (36)$$

Hence, equipping the external algebra of the Kähler manifold by a pair of antibrackets (corresponding to the choice  $\alpha = 0, \pi/2$ ) one can describe the external calculus in terms of the pair of antibrackets. One can observe that  $\{G + \mathcal{F}, \}_1$

defines equivariant differential, while the function  $G - \mathcal{F}$  defines the equivariant Chern class; the relation (33) corresponds to the well-known Lie identity  $di_X + i_X d = L_X$ , and so on. The Lie derivative of odd symplectic structure along the vector field  $\{G + \mathcal{F}, \}_1$  yields the equivariant even pre-symplectic structure, generating equivariant Euler classes of the underlying Kähler manifold [15].

Let us mention also another similarity between the system under consideration and the Lagrangian BRST quantization schemes. The Batalin-Vilkovisky formalism admits the BRST-antiBRST-invariant extension which is known, in its most general form, under the name of “triplectic formalism” [18]. This extension is formulated by the use of a pair of antibrackets,  $\{, \}_1^\alpha$ , the pair of nilpotent odd vector fields obeying the compatibility conditions

$$\begin{aligned} & (-1)^{(p(f)+1)(p(h)+1)} \{f, g\}_1^\alpha, h\}_1^\beta + \\ & \quad + \text{cicl.perm. } f, g, h = 0 \\ & V^{\{\alpha} \{f, g\}_1^\beta\} = \\ & = \{V^{\{\alpha} f, g\}_1^\beta\} + (-1)^{p(f)+1} \{f, V^{\{\alpha} g\}_1^\beta\} \\ & V^{\{\alpha} V^\beta\} = 0. \end{aligned} \quad (37)$$

It is easy to see, that the antibrackets (30) and the vector fields  $V = \{\mathcal{F}, \}_1$  corresponding to the choice  $\alpha = 1, i$ , form a “classical triplectic algebra” (37). However, there is a crucial difference between the triplectic algebra arising in the context of BRST quantization and the above-presented one: the antibrackets appearing in the triplectic formalism, are degenerate, while the above-presented ones are *nondegenerate*. On the other hand, “classical triplectic algebra” corresponding to degenerate antibrackets, is also related with Kähler geometry [19].

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